Ex: Compute the line integral $\int_C f\,ds$ for $f(x,y,z) = x^2 y$

and C the curve parametrized by $\vec{r}(t) = \langle t, \cos(2t), \sin(2t) \rangle$ on $0 \le t \le 2\pi$.

Sol: $\int_C f\,ds = \int_{t=a}^{b} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

$a = 0, \quad b = 2\pi$

$f(\vec{r}(t)) = t^2 \cos(2t)$

$\vec{r}\,'(t) = \langle 1, -2\sin(2t), 2\cos(2t) \rangle$

$|\vec{r}\,'(t)| = \sqrt{1 + 4\sin^2(2t) + 4\cos^2(2t)} = \sqrt{5}$

$= \int_{t=0}^{2\pi} t^2 \cos(2t) \cdot \sqrt{5}\,dt$ ← looks like we should use integration-by-parts. let's compute the general antiderivative...

Note:

IBP: $\int t^2 \cos(2t)\,dt$ $\quad \begin{pmatrix} u_1 = t^2 \\ du_1 = 2t\,dt \end{pmatrix}$ $\quad dv_1 = \cos(2t)\,dt$, $v_1 = \frac{1}{2}\sin(2t)$

$= \int u_1\,dv_1 = u_1 v_1 - \int v_1\,du_1$

$= \frac{1}{2}t^2 \sin(2t) - \int t \sin(2t)\,dt$ $\quad \begin{pmatrix} u_2 = t \\ du_2 = dt \end{pmatrix}$ $\quad dv_2 = \sin(2t)\,dt$, $v_2 = -\frac{1}{2}\cos(2t)$

IBP: $= \frac{1}{2}t^2 \sin(2t) - \int u_2\,dv_2 = \frac{1}{2}t^2 \sin(2t) - \left(u_2 v_2 - \int v_2\,du_2\right)$

$= \frac{1}{2}t^2 \sin(2t) + \frac{1}{2}t\cos(2t) + \int -\frac{1}{2}\cos(2t)\,dt$

$= \frac{1}{2}t^2 \sin(2t) + \frac{1}{2}t\cos(2t) - \frac{1}{4}\sin(2t) + C.$

$\therefore \int_C f\,ds = \int_{t=0}^{2\pi} t^2 \cos(2t) \cdot \sqrt{5}\,dt$

$= \sqrt{5}\left[\frac{1}{2}t^2 \sin(2t) + \frac{1}{2}t\cos(2t) - \frac{1}{4}\sin(2t)\right]_{t=0}^{2\pi}$

$= \sqrt{5}\left(\left(0 + \frac{1}{2}\cdot 2\pi \cdot 1 - 0\right) - (0 + 0 - 0)\right)$

$= \sqrt{5}\pi$